

# The Einstein-like field theory and the dislocations with finite-sized core

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## Abstract

Einstein-like Lagrangian field theory is developed to describe elastic solid containing dislocations with finite-sized core. The framework of the Riemann–Cartan geometry in three dimensions is used, and the core self-energy is expressed by the translational part of the general Lagrangian quadratic in torsion and curvature. In the Hilbert–Einstein case, the gauge equation plays the role of non-conventional incompatibility law. The stress tensor of the modified screw dislocations is smoothed out within the core. The renormalization of the elastic constants caused by proliferation of the dislocation dipoles is considered. The use of the singularityless dislocation solution modifies the renormalization of the shear modulus in comparison with the case of singular dislocations.

**Keywords:** Translational gauging; Hilbert-Einstein gauge equation; Screw dislocation; Shear modulus.

# 1 Introduction

Considerable role of geometry in modern theoretical physics is widely acknowledged. The differential geometry provides a framework for analogies between the low-dimensional gravity and the defects in solids [1–7]. Geometric considerations are of importance for investigation of interplay between such subjects as the statistical physics of defects, symmetries, and phase transitions [8–10]. The multivalued fields are of great importance in the condensed matter physics for description of defects and phase transitions, [3]. Multivaluedness of the transformation functions (of the displacement field in the case of dislocated crystals) is responsible for the topological non-triviality of the line-like defects. The point singularities of these functions constitute the densities of the line-like defects. The multivalued coordinate transformations (corresponding to the defected crystal) transform a flat space into general affine spaces with curvature and torsion. It is the Riemann-Cartan geometry of the manifolds with curvature and torsion which is relevant to the theory of solids with dislocations and disclinations, [1–7].

An original gauge approach to the statistical physics of the line-like defects (of dislocations and disclinations, in particular) is summarized in the monographs [1–3]. The dislocation core energy is proposed in [2] in order to formulate the lattice model of the defect melting and the corresponding disorder field theory. The dislocation core energy in [2] depends quadratically on the defect tensor which is expressed through the disclination and dislocation densities.

The singular character of the dislocation solutions of incompatible elasticity theory is an idealization since the dislocation core is not captured by the classical elasticity. The Lagrangian gauge approaches proposed in [11–14] enable to describe the dislocations possessing the cores of finite size.<sup>1</sup> The gauge Lagrangians are chosen in [11–13] in the form quadratic in the gauge field strength, i.e., in the dislocation density. The approach of the Einstein-like field theory [14, 15] is based on the Hilbert–Einstein gauge Lagrangian for description of the core self-energy. Remind that the geometric theory of defects proposed in [4] is based on the most general eight-parameter gauge Lagrangian invariant with respect of localized action of three-dimensional Euclidean group. Since the dislocation density is identified as the differential-geometric torsion, [2, 3, 7], the gauge Lagrangians [11–15] may be viewed as parts of the gauge Lagrangian [4].

In the case of the screw dislocations, the Einstein-type gauge equation plays the role of non-conventional incompatibility law. The model [15] allows for a continuation of the stresses of the screw dislocation within the core so that artificial cut-off does not occur both in linear and quadratic approximations. The dislocation cross-section is no longer “point-wise”, while sufficiently far from the core region the dislocation demonstrates its topological nature.

A thermodynamical description is developed in [18, 19] for non-singular screw dislocations described by the Einstein-like field theory [15]. The dislocation core energy in [18, 19] originates, by means of linearizations, from the Lagrangian [14]. The partition function of collection of the screw dislocations is proposed in [18] in the functional

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<sup>1</sup> An interest to non-singular strings, [16], and fluxes of finite width, [17], also deserves a mention.

integral form so that the modified screw dislocation arises as its saddle point. Eventually, the effective energy of the system of positive and negative modified dislocations along an elastic cylinder is that of two-component two-dimensional Coulomb gas of particles with smoothed out coupling.<sup>2</sup>

Proliferation of the dislocation dipoles is responsible for the renormalization of the elastic constants, [20–23]. From the viewpoint of nano-physics it is attractive, [24, 25], to study the properties of the elastic moduli using the modified dislocation solutions [11–15, 26]. In Refs. [18, 19] the renormalization of the elastic constants due to proliferation of dipoles of the modified screw dislocations is studied. The renormalization of the shear modulus under the influence of the dislocation cores is obtained near the melting transition [19].

## 2 The dislocations with finite-sized core

Initial and deformed states of the dislocated three-dimensional solid are described by squared length elements  $g_{ij}dx^i dx^j$  and  $\eta_{ab}d\xi^a d\xi^b$ , provided that the map  $\mathbf{x} \mapsto \boldsymbol{\xi}(\mathbf{x})$  transforms initial state to the deformed one. The approach of [15] is based on the Eulerian picture expressed by the strain tensor  $e_{ab}$  referred to the deformed state:

$$\eta_{ab}d\xi^a d\xi^b - g_{ij}dx^i dx^j = 2e_{ab}d\xi^a d\xi^b, \quad (1)$$

where

$$2e_{ab} \equiv \eta_{ab} - g_{ab}, \quad g_{ab} \equiv g_{ij}\mathcal{B}_a^i \mathcal{B}_b^j. \quad (2)$$

The metric  $g_{ab}$  (2) is the Cauchy deformation tensor, and the components  $\mathcal{B}_a^i$  are given by one-forms  $dx^i = \mathcal{B}_a^i d\xi^a$ . Assume that  $\mathcal{B}_a^i$  are  $\mathbb{T}(3)$ -gauged:

$$\mathcal{B}_a^i = \frac{\partial x^i}{\partial \xi^a} - \varphi_a^i. \quad (3)$$

Then, dislocations are allowed provided that one-forms  $\mathcal{B}_a^i d\xi^a$  are not globally closed due to the gauge potentials  $\varphi_a^i$ . The entries  $\varphi_a^i$  are the translational gauge potentials, which behave under the local shifts  $x^i \rightarrow x^i + \eta^i(x)$  as follows:

$$\begin{aligned} \frac{\partial x^i}{\partial \xi^a} &\longrightarrow \frac{\partial x^j}{\partial \xi^a} \left( \delta_j^i + \frac{\partial \eta^i}{\partial x^j} \right), \\ \varphi_a^i &\longrightarrow \varphi_a^i + \frac{\partial x^j}{\partial \xi^a} \frac{\partial \eta^i}{\partial x^j}. \end{aligned} \quad (4)$$

The transformations (4) ensure the gauge invariance of  $\mathcal{B}_a^i$  (3). Non-triviality of the core for the screw dislocation should be associated with the gauge fields  $\boldsymbol{\varphi} \equiv (\varphi_a^i)$ .

The Lagrangian of the model includes, apart from elastic energy, the translational part of the eight-parameter Lagrangian  $\mathcal{L}_g(\omega, \mathcal{B})$ , which is invariant under the

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<sup>2</sup>The approach [18, 19] is concerned with individual defects, and it is not of the type of the disorder field theories developed in [1–3] for various phase transitions (e.g., superfluidity, superconductivity, melting).

coordinate shifts and local rotations [4]. The translational part of  $\mathcal{L}_g$  is given by three independent invariants quadratic in the torsion tensor components (identified as the dislocation density components)  $\mathcal{T}_{ab}^c = (\partial_a \mathcal{B}_b^i - \partial_b \mathcal{B}_a^i) B_i^c$  (here  $B_j^c$  are reciprocals of  $\mathcal{B}_a^i$ ):

$$\mathcal{B}^{-1} \mathcal{L}_g|_{\omega=0} = -\frac{1}{4} \mathcal{T}_{abc} (\beta_1 \mathcal{T}^{abc} + \beta_2 \mathcal{T}^{cab} + \beta_3 \mathcal{T}^{eb} \eta^{ac}), \quad \mathcal{B} \equiv \det \mathcal{B}_a^i. \quad (5)$$

The model is governed by the Hilbert–Einstein gauge equation,

$$G^{ef} = \frac{1}{2\ell} (\sigma^{ef} - (\sigma_{\text{bg}})^{ef}), \quad (6)$$

provided that  $\beta_1 = -\ell$ ,  $\beta_2 = 2\ell$ ,  $\beta_3 = 4\ell$  (the, so-called, Hilbert–Einstein parametrization, [4]). Left-hand side of (6) is given by the Einstein tensor  $G^{ef} \equiv \frac{1}{4} \mathcal{E}^{eab} \mathcal{E}^{fcd} R_{abcd}$ , where  $\mathcal{E}^{abc}$  is the Levi-Civita tensor, and  $R_{abc}^d$  is the Riemann–Christoffel tensor calculated for the metric  $g_{ab}$  (2). The driving source in right-hand side of (6) is expressed by the difference  $\sigma^{ef} - (\sigma_{\text{bg}})^{ef}$ , where  $\sigma^{ef}$  and  $(\sigma_{\text{bg}})^{ef}$  are the stress tensors corresponding to, so-called, total and *background* contributions. The field  $\sigma_{\text{bg}} \equiv (\sigma_{\text{bg}})^{ef}$  is determined by a prescribed distribution of the background dislocations. Besides,  $\ell$  is the energy scale of the gauge field  $\varphi$ . The equilibrium equations are:  $\overset{(\eta)}{\nabla}_a \sigma^{ab} = 0$ ,  $\overset{(\eta)}{\nabla}_a (\sigma_{\text{bg}})^{ab} = 0$ , where  $\overset{(\eta)}{\nabla}_a$  is the covariant derivative with respect to  $\eta_{ab}$ .

The *modified* (i.e., non-singular) screw dislocation arises in T(3)-gauge model proposed in [15]. In the first order, the stress–strain relation is given by  $\overset{(1)}{\sigma}_{\phi z} = 2\mu \overset{(1)}{e}_{\phi z}$ , while stress field of the first-order,  $\overset{(1)}{\sigma}_{\phi z}$ , takes the form in the cylindrical coordinates:

$$\overset{(1)}{\sigma}_{\phi z} = -\mu \overset{(1)}{\partial}_\rho \phi = \frac{b\mu}{2\pi} \frac{1}{\rho} (1 - \kappa \rho K_1(\kappa \rho)), \quad \overset{(1)}{\phi} \equiv \frac{-b}{2\pi} (\log \rho + K_0(\kappa \rho)). \quad (7)$$

Here,  $\overset{(1)}{\phi}$  is the stress potential of the modified dislocation,  $\mu$  is the shear modulus, the Burgers vector component along  $z$ -axis is  $b_z = b$ , and  $\kappa = (\mu/\ell)^{\frac{1}{2}}$ . The asymptotical behavior of the stress at  $\kappa \rho \gg 1$  is given by the background contribution

$$(\overset{(1)}{\sigma}_{\text{bg}})_{\phi z} = \frac{b\mu}{2\pi} \frac{1}{\rho}.$$

The dislocation core corresponds to  $\rho \lesssim 1/\kappa$ , since the gauge correction to  $\frac{1}{\rho}$  is exponentially small outside the core. Inside the core,  $\overset{(1)}{\sigma}_{\phi z}$  behaves as  $A\rho \log(B\rho)$  at  $\rho \rightarrow 0$ .

In the second order, the stress components  $\overset{(2)}{\sigma}_{\rho\rho}$ ,  $\overset{(2)}{\sigma}_{\phi\phi}$ ,  $\overset{(2)}{\sigma}_{zz}$  differ from zero, [15]. For instance, sufficiently far from the core we obtain:

$$\begin{aligned} \overset{(2)}{\sigma}_{\rho\rho} \Big|_{\kappa\rho \gg 1} &\approx \frac{2\tilde{n}}{\rho^2} \left( \log \frac{\rho}{\rho_1} - \left( \frac{\rho}{R} \right)^2 \log \frac{R}{\rho_1} \right), \\ \overset{(2)}{\sigma}_{\phi\phi} \Big|_{\kappa\rho \gg 1} &\approx \frac{2\tilde{n}}{\rho^2} \left( 1 - \log \frac{\rho}{\rho_1} - \left( \frac{\rho}{R} \right)^2 \log \frac{R}{\rho_1} \right), \end{aligned} \quad (8)$$

where  $\tilde{n} \sim \mu b^2$ , and  $R$  is the cylinder's external radius. The asymptotical expressions (8) coincide with the answers arising in the conventional approach based on the quadratic elasticity theory although they include the length  $\rho_1$  instead of an artificial cut-off radius  $\rho_c$ . The components  $\sigma_{\rho\rho}^{(2)}$ ,  $\sigma_{\phi\phi}^{(2)}$  behave as  $o(\rho^2 \log^3 \rho)$  at  $\rho \rightarrow 0$ . The stress  $\sigma_{zz}^{(2)}$  also looks classically at large distances though its long-ranged expression is parametrized by another length  $\rho_m$  instead of  $\rho_c$ :

$$\sigma_{zz}^{(2)} \Big|_{\kappa\rho \gg 1} \approx 2 \left( \nu \tilde{n} + \left( \frac{b}{2\pi} \right)^2 \mu^3 (1 + \nu) C_3 \right) \left( \frac{1}{\rho^2} - \frac{2}{R^2} \log \frac{R}{\rho_m} \right),$$

where  $\nu$  is the Poisson ratio (see [15] for the elastic constant  $C_3$ ). In the gauge approach [15], the radii  $\rho_1$  and  $\rho_m$  are expressed through the elastic constants of second and third orders. The appearance of several lengths demonstrates the defect core as a radially layered region without a single sharp boundary. The continuation of the stresses is due to the gauge equation (6) considered as the non-conventional incompatibility law.

### 3 The shear modulus renormalization

The approach of [15] to description of singularityless dislocations is elaborated in [18] further for studying the collection of the modified screw dislocations as a thermodynamic ensemble. Specifically, a long enough elastic cylinder pierced by non-singular screw dislocations is studied. The corresponding partition function  $\mathcal{Z}$  is given by the functional integral [18]:

$$\mathcal{Z} = \int e^{\beta \mathcal{L}} [\text{Meas}], \quad \mathcal{L} \equiv \mathcal{L}_{\text{el}} + \mathcal{L}_{\text{core}} - i\mathcal{E}_{\text{ext}}, \quad (9)$$

where  $\beta$  is inverse of the absolute temperature, and  $[\text{Meas}]$  is appropriately normalized integration measure. In the framework of the elasticity plane problem,  $\mathcal{L}$  (9) includes the standard Lagrangian of linear elasticity

$$\mathcal{L}_{\text{el}} = \frac{-1}{2\mu} \int (\sigma_i^b + \sigma_i^c)^2 d^2x, \quad (10)$$

while the other contributions are:

$$\mathcal{L}_{\text{core}} = \int (\ell (\partial_i e_j - \partial_j e_i)^2 + 2e_i \sigma_i^c) d^2x, \quad \mathcal{E}_{\text{ext}} = \int \sigma_i^b (\partial_i u - 2\mathcal{P}_i) d^2x, \quad (11)$$

where  $u \equiv u_3$  and  $e_i \equiv e_{i3}$  are displacement and total strain components ( $i = 1, 2$ , and  $Oz \equiv Ox_3$  is the cylinder axis). Here we use the notations  $\sigma_i^\# \equiv \sigma_{i3}^\#$  for the stress components which are independent integration variables corresponding to, so-called, *background* ( $\#$  is b) or *core* ( $\#$  is c) contributions. The non-conventional stress  $\sigma_i^c$  is short-ranged, and therefore it modifies the background one,  $\sigma_i^b$ , within the core of the modified dislocation, [15]. The Lagrangian  $\mathcal{L}_{\text{core}}$  (11) originates from the Lagrangian

(5). Besides,  $\mathcal{P}_i$  in  $\mathcal{E}_{\text{ext}}$  (11) is the plastic source which is concentrated on cut surfaces bounded by the dislocation lines. It specifies distribution of the background defects.

The functional integral (9) is estimated in [18] by means of steepest descent. The stress potential of array of the modified dislocations (7) plays the role of the corresponding saddle point because of the choice of the functional  $\mathcal{L}$  expressed by (9), (10), and (11) (with  $\mathcal{P}_i$  substituted appropriately).

Therefore the estimate of  $\mathcal{Z}$  (9) leads us to the effective energy  $\mathcal{W} = \frac{-1}{\beta} \log \mathcal{Z}$  of  $2\mathcal{N}$  modified screw dislocations lying along the cylinder axis with unit Burgers vectors ( $b = 1$ ) [18]:

$$\mathcal{W} = \frac{-\mu}{4\pi} \sum_{I,J} (\mathcal{U}(\kappa|\mathbf{y}_I^+ - \mathbf{y}_J^+|) + \mathcal{U}(\kappa|\mathbf{y}_I^- - \mathbf{y}_J^-|) - 2\mathcal{U}(\kappa|\mathbf{y}_I^+ - \mathbf{y}_J^-|)), \quad (12)$$

$$\mathcal{U}(s) \equiv \log\left(\frac{\gamma}{2}s\right) + K_0(s). \quad (13)$$

The energy  $\mathcal{W}$  (12) demonstrates that the array of the modified screw dislocations is equivalent to the two-dimensional Coulomb gas of unit charges  $\pm 1$  characterized by the two-body potential  $\mathcal{U}$  (13) which is logarithmic at large separation but tends to zero for the charges sufficiently close to each other. Due to condition of “electroneutrality”, the number of positive dislocations at the points  $\{\mathbf{y}_I^+\}_{1 \leq I \leq \mathcal{N}}$  is equal to the number of negative ones at the points  $\{\mathbf{y}_I^-\}_{1 \leq I \leq \mathcal{N}}$ .

We consider the grand-canonical ensemble of the dislocations in the *dipole phase*, which corresponds to bound pairs of the dislocations with opposite Burgers vectors. Define two-point stress-stress correlation function:

$$\langle\langle \sigma_i(\mathbf{x}_1) \sigma_j(\mathbf{x}_2) \rangle\rangle = \mathbf{Z}_{\text{dip}}^{-1} \sum_{\mathbf{n} \& \mathbf{p}} \int \sigma_i(\mathbf{x}_1) \sigma_j(\mathbf{x}_2) e^{\beta \mathcal{L}} [\text{Meas}], \quad (14)$$

where  $\sigma_i(\mathbf{x}) = \sigma_i^b(\mathbf{x}) + \sigma_i^c(\mathbf{x})$  is the total stress distribution,  $\mathcal{L}$  is expressed by (9), (10), (11), and  $\mathbf{Z}_{\text{dip}}$  is the partition function in the dipole approximation. Besides,  $\sum_{\mathbf{n} \& \mathbf{p}}$  is summation over number of dipoles and over their positions. The functional integration in (14) is defined with respect of a given distribution of the dislocation lines expressed by  $\mathcal{P}_i$ .

The grand-canonical partition function  $\mathbf{Z}_{\text{dip}}$  of the Coulomb system described by the energy (12) takes the form in the dipole phase [18]:

$$\begin{aligned} \mathbf{Z}_{\text{dip}} &= \sum_{\mathcal{N}=0}^{\infty} \frac{1}{\mathcal{N}!} \prod_{I=1}^{\mathcal{N}} \int d^2 \boldsymbol{\xi}_I \int d^2 \boldsymbol{\eta}_I \exp[-2\beta \mathcal{N} \Lambda - \beta \mathcal{W}_{\text{dip}}], \\ \mathcal{W}_{\text{dip}} &\equiv \sum_{I=1}^{\mathcal{N}} w(\boldsymbol{\eta}_I) + \sum_{I < J} w_{IJ}, \quad \beta w(\boldsymbol{\eta}) \equiv \mathcal{K} \mathcal{U}(\kappa \boldsymbol{\eta}), \end{aligned} \quad (15)$$

where  $\beta = \frac{1}{T}$  is inverse temperature (the Boltzmann constant is unity),  $\mathcal{N}$  is the number of dipoles,  $\Lambda$  is the chemical potential per dislocation, and  $\mathcal{K} \equiv \frac{\mu\beta}{2\pi}$ . Besides,  $w(\boldsymbol{\eta}_I)$  is the energy of  $I^{\text{th}}$  dipole centered in  $\boldsymbol{\xi}_I = (\mathbf{y}_I^+ + \mathbf{y}_I^-)/2$  with the dipole momentum  $\boldsymbol{\eta}_I = \mathbf{y}_I^+ - \mathbf{y}_I^-$ . The integration goes over the cylinder’s cross-section. The

energy  $\mathcal{W}_{\text{dip}}$  in (15) arises, [18], in the dipole approximation from (12). The energy of interaction between  $I^{\text{th}}$  and  $J^{\text{th}}$  dipoles  $w_{IJ}$  in  $\mathcal{W}_{\text{dip}}$  (15) is of the form:

$$\beta w_{IJ} \equiv -\mathcal{K}(\boldsymbol{\eta}_I, \boldsymbol{\partial}_{\boldsymbol{\xi}_I})(\boldsymbol{\eta}_J, \boldsymbol{\partial}_{\boldsymbol{\xi}_J}) \mathcal{U}(\kappa|\boldsymbol{\xi}_I - \boldsymbol{\xi}_J|), \quad (16)$$

where  $\boldsymbol{\partial}_{\boldsymbol{\xi}}$  implies 2-vector  $(\partial_{\xi_1}, \partial_{\xi_2}) \equiv (\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$  and  $(\boldsymbol{\eta}_I, \boldsymbol{\partial}_{\boldsymbol{\xi}_I})$  is the scalar product of 2-vectors  $\boldsymbol{\eta}_I$  and  $\boldsymbol{\partial}_{\boldsymbol{\xi}_I}$ .

The integral in right-hand side of (14) is calculated in [18], and the correlator in the dipole representation of the Coulomb gas takes the form:

$$\begin{aligned} \langle\langle \sigma_i(\mathbf{x}_1) \sigma_j(\mathbf{x}_2) \rangle\rangle &= \frac{-\mu}{2\pi\beta} \partial_{(\mathbf{x}_1)_i} \partial_{(\mathbf{x}_2)_j} \mathcal{U}(\kappa|\mathbf{x}_1 - \mathbf{x}_2|) \\ &+ \mathbf{Z}_{\text{dip}}^{-1} \sum_{\mathbf{n} \& \mathbf{p}} \sigma_i(\mathbf{x}_1) \sigma_j(\mathbf{x}_2) e^{-\beta \mathcal{W}_{\text{dip}}}, \end{aligned} \quad (17)$$

where  $\sigma_i = \mu \epsilon_{ik} \partial_{x_k} \phi$ , and  $\phi \equiv \overset{(1)}{\phi}$  is the stress potential (compare with (7)) of superposition of  $\mathcal{N}$  dipoles. Positions of dipoles are confined within a disk of radius  $R$ .

We consider the renormalization of the shear modulus  $\mu$  caused by proliferation of the dislocation dipoles. The renormalized  $\mu_{\text{ren}}$  is defined as follows:

$$\frac{1}{\mu_{\text{ren}}} \equiv \frac{\beta}{\mu^2 \mathcal{S}} \sum_{i,k=1,2} \iint \langle\langle \sigma_i(\mathbf{x}_1) \sigma_k(\mathbf{x}_2) \rangle\rangle d^2 \mathbf{x}_1 d^2 \mathbf{x}_2, \quad (18)$$

where  $\mathcal{S}$  is the cross-section area. We approximately obtain from (17) the stress-stress correlation function:

$$\begin{aligned} \langle\langle \sigma_i(\mathbf{x}_1) \sigma_j(\mathbf{x}_2) \rangle\rangle &\approx \frac{-\mu}{2\pi\beta} \left( \partial_{(\mathbf{x}_1)_i} \partial_{(\mathbf{x}_2)_j} \mathcal{U}(\kappa|\Delta \mathbf{x}|) \right. \\ &\left. - \sum_{\tilde{n}=1}^{\infty} (-\beta \mu d)^{\tilde{n}} (\epsilon_{ik} \epsilon_{jl} \partial_{(\mathbf{x}_1)_k} \partial_{(\mathbf{x}_2)_l}) \left[ \mathcal{U}(\kappa|\Delta \mathbf{x}|) + \frac{\kappa|\Delta \mathbf{x}|}{2} K_1(\kappa|\Delta \mathbf{x}|) \right] \right). \end{aligned} \quad (19)$$

The term given by  $\tilde{n} = 1$  in right-hand side of (19) coincides with that obtained in [18] in the case of non-interacting dipoles. The contributions at  $\tilde{n} \geq 2$  in right-hand side of (19) are due to the dipole-dipole coupling (16). One uses (18), (19) and obtains:

$$\frac{1}{\mu_{\text{ren}}} = \frac{1}{\mu} \mathcal{C}_1(\kappa R) + \frac{\beta d}{1 + \mu \beta d} \mathcal{C}_2(\kappa R), \quad (20)$$

where  $d$  is proportional to mean area covered by the dipoles (more precisely,  $\beta \mu d \equiv \pi \mathcal{K} \langle \boldsymbol{\eta}^2 \rangle \bar{N}$ , where  $\langle \boldsymbol{\eta}^2 \rangle$  is mean square of the dipole momentum, and  $\bar{N}$  is average dipole density, [18]). The functions  $\mathcal{C}_1(\kappa R)$  and  $\mathcal{C}_2(\kappa R)$  are given by the modified Bessel functions:

$$\begin{aligned} \mathcal{C}_1(\kappa R) &= 1 - 2K_1(\kappa R)I_1(\kappa R), \\ \mathcal{C}_2(\kappa R) &= 2 - 2I_1(\kappa R)(K_1(\kappa R) - \kappa R K_1'(\kappa R)), \end{aligned}$$

and  $K'_1(z) = \frac{d}{dz}K_1(z)$ . The influence of the cores causes the size-dependence of the functions  $\mathcal{C}_1(\kappa R)$  and  $\mathcal{C}_2(\kappa R)$ . Equation (20) demonstrates that the shear modulus  $\mu_{\text{ren}}$  depends on the dimensionless ratio  $R/\kappa^{-1} = \kappa R$  of two lengths characterizing the sample's cross-section and the dislocation core sizes. The coefficients  $\mathcal{C}_1(\kappa R)$  and  $\mathcal{C}_2(\kappa R)$  both are positive and less than unity though tend to unity at  $\kappa R \rightarrow \infty$ :

$$\mathcal{C}_1(\kappa R) \approx 1 - \frac{1}{\kappa R} + \dots, \quad \mathcal{C}_2(\kappa R) \approx 1 - \frac{3}{2\kappa R} + \dots,$$

where the ellipsis imply the terms  $\mathcal{O}((\kappa R)^{-2})$ .

The dependence on the size-parameter  $\kappa R$  in Eqs. (20) displays the effect of the non-conventional dislocation solution on the shear modulus near the melting transition. As it has been experimentally confirmed in [24], properly normalized Young modulus tends to the universal value  $16\pi$  at  $T \rightarrow T_c^-$  (see [2]). The present approach demonstrates that the renormalized shear modulus deviates from a multiple of  $\pi$  due to the smoothed out character of the dislocations:

$$\frac{\mu_{\text{ren}}(T_c^-)}{T_c} \approx \frac{8\pi}{\mathcal{C}_1(\kappa R)} \xrightarrow{\kappa R \gg 1} 8\pi, \quad d \ll 1. \quad (21)$$

## 4 Discussion

The results obtained should be applicable to nanotubes and nanowires with comparable  $R$  and  $\kappa^{-1}$ . With regard at the results of [24, 27] concerning the colloidal crystals, it is hopeful that experimental nanophysics could provide us opportunities of verification of the relations of the type of Eqs. (20) and (21) near the melting point.<sup>3</sup> The colloidal crystals have also been mentioned in [25] among other candidates for observing the effects due to the elastic constants renormalization. Further studies of the effects due to the singularityless character of the dislocations look attractive as far as nano-materials and nano-physics are concerned.

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## References

- [1] H. Kleinert, *Gauge Fields in Condensed Matter*. Vol. I (World Scientific, Singapore, 1989).
- [2] H. Kleinert, *Gauge Fields in Condensed Matter*. Vol. II (World Scientific, Singapore, 1989).
- [3] H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (World Scientific, Singapore, 2008).

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<sup>3</sup>Defect mediated melting attracts attention in various systems, [28–30].



- [4] M. O. Katanaev, I. V. Volovich, *Ann. Phys.* **216**, 1 (1992).
- [5] M. O. Katanaev, *Physics–Uspekhi* **48**, 675 (2005).
- [6] G. de Berredo-Peixoto, M. O. Katanaev, *J. Math. Phys.* **50**, 042501 (2009).
- [7] M. O. Katanaev, *Geometrical Methods in Mathematical Physics*, *arXiv:1311.0733v3 [math-ph]*.
- [8] D. R. Nelson, *Defects and Geometry in Condensed Matter Physics* (CUP, Cambridge, 2002).
- [9] Gil Young Cho, O. Parrikar, Yizhi You, R. G. Leigh, T. L. Hughes, *Phys. Rev. B* **91**, 035122 (2015).
- [10] Ke Liu, J. Nissinen, Z. Nussinov, R.-J. Slager, Kai Wu, J. Zaanen, *Phys. Rev. B* **91**, 075103 (2015).
- [11] M. C. Valsakumar, D. Sahoo, *Bull. Mater. Sci.* **10**, 3 (1988).
- [12] D. G. B. Edelen, *Int. J. Engng Sci.* **34**, 81 (1996).
- [13] M. Lazar, *J. Phys. A: Math. Gen.* **35**, 1983 (2002).
- [14] C. Malyshev, *Ann. Phys. (NY)* **286**, 249 (2000).
- [15] C. Malyshev, *J. Phys. A: Math. Theor.* **40**, 10657 (2007).
- [16] M. Caselle, P. Grinza, N. Magnoli, *J. Stat. Mech.* **0611**, P11003 (2006).
- [17] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, H. H. Harutyunyan, *Phys. Rev. D* **91**, 064034 (2015).
- [18] C. Malyshev, *J. Phys. A: Math. Theor.* **44**, 285003 (2011).
- [19] C. Malyshev, *Ann. Phys. (NY)* **351**, 22 (2014).
- [20] A. Holz, J. T. N. Medeiros, *Phys. Rev. B* **17**, 1161 (1978).
- [21] D. R. Nelson, *Phys. Rev. B* **18**, 2318 (1978).
- [22] D. R. Nelson, B. I. Halperin, *Phys. Rev. B* **19**, 2457 (1979).
- [23] A. P. Young, *Phys. Rev. B* **19**, 1855 (1979).
- [24] H. H. von Grünberg, P. Keim, K. Zahn, G. Maret, *Phys. Rev. Lett.* **93**, 255703 (2004).
- [25] S. Panyukov, Y. Rabin, *Phys. Rev. B* **59**, 13657 (1999-I).
- [26] M. Yu. Gutkin, E. C. Aifantis, *Scripta Mater.* **40**, 559 (1999).

- [27] S. Deutschländer, A. M. Puertas, G. Maret, P. Keim, *Phys. Rev. Lett.* **113**, 127801 (2014).
- [28] S. A. Gifford, G. Baym, *Phys. Rev. A* **78**, 043607 (2008).
- [29] T. Saiki, R. Ikeda, *Phys. Rev. B* **83**, 174501 (2011).
- [30] H. Kleinert, *Europhys. Lett.* **102**, 56002 (2013).